A MODEL OF THE PLANNING, PROGRAMMING, AND BUDGETING PROBLEM

by

Robert Stephen Brent

United States
Naval Postgraduate School

THESIS

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Robert Stephen Brent
Lieutenant (junior grade), Supply Corps
United States Navy
B.A., Duke University, 1969

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ABSTRACT

A model of the planning, programming, and budgeting problem is formulated. The variables of the model are resources, elements, characteristics, benefits (measures of effectiveness), and costs. The nature of the PPB problem requires that the model incorporate multiple measures of benefit and cost. To characterize efficient choices in the PPB context decision rules which are necessary and sufficient for efficiency are derived. Discounting of benefits over time is discussed. Sensitivity analysis of the model is performed. Decentralization possibilities in the model are explored.

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TABLE OF SYMBOLS

E_l^t	amount of l^{th} measure of effectiveness in time period t
C_m	magnitude of m^{th} cost measure
$C_m(\ )$	m^{th} cost function
z_i^t	amount of i^{th} characteristic in time period t
y_j^t	amount of j^{th} element in time period t
x_k^t	amount of k^{th} resource in time period t
$\underline{\underline{B}}^t$	benefit technology matrix in time period t
$\underline{\underline{B}}^{(E)t}$	upper part of benefit technology matrix in time period t
$\underline{\underline{B}}^{(z)t}$	lower part of benefit technology matrix in time period t
$\underline{\underline{S}}^t$	systems technology matrix in time period t
$\underline{\underline{S}}^{(z)t}$	upper part of systems technology matrix in time period t
$\underline{\underline{S}}^{(y)t}$	lower part of systems technology matrix in time period t
$\underline{\underline{P}}^t$	production technology matrix in time period t
$\underline{\underline{P}}^{(y)t}$	upper part of production technology matrix in time period t
$\underline{\underline{P}}^{(x)t}$	lower part of production technology matrix in time period t
η_a^t	activity level of a^{th} benefit activity in time period t
ζ_b^t	activity level of b^{th} systems activity in time period t
ω_g^t	activity level of g^{th} production activity in time period t
ϕ_l^t	weighting of l^{th} benefit in time period t

Ψ_m weighting of m^{th} cost measure

$\lambda_l^{(E)t}$ Lagrange multiplier of l^{th} benefit in time period t

$\lambda_m^{(C)}$ Lagrange multiplier of m^{th} cost measure

$\lambda_i^{(z)t}$ Lagrange multiplier of i^{th} characteristic in time period t

$\lambda_j^{(L)t}$ Lagrange multiplier of j^{th} elements in time period t

d_i marginal rate of psychological discount in time period i

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I. INTRODUCTION

The planning, programming, and budgeting problem for a governmental agency involves choice among alternatives in the face of scarce resources. It can therefore be examined using the principles of economics, since that subject is fundamentally concerned with the problem of choice in allocating scarce resources. In this paper a mathematical model of the PPB problem is presented. The model is similar in structure to several economic models, particularly certain models of consumer choice. Like those models in economics, the purpose of this model at its present stage of development is not to reproduce the real world nor give numerical answers to a decision-maker. Rather the purpose of this model is to develop a conceptual framework for the PPB problem, using mathematics to ensure a logical development.

The purpose of a PPB system is to decide how to allocate the scarce resources of the Department of Defense. Different allocations of resources are referred to as "alternatives" in this paper; alternatives can be viewed more generally as different ways of structuring the Department of Defense forces.

The PPB process may be viewed conceptually as an iterative process, similar to the Walrasian tâtonnement

sometimes used in economics to explain how a market works.[1] In tâtonnement an auctioneer sends out prospective prices to producers and consumers, which he adjusts until supply equals demand. In a PPB system the central planners correspond to the auctioneer, the agents to the producers, the decision-maker to the consumers, and the prospective indices to prices. The agents are commanders of areas of the Department of Defense, such as ASW or armor. The decision-maker is usually the Secretary of Defense. The central planners send prospective indices, such as manpower limits, major mission budgets, and illustrative forces to the agents. The agents know the technological possibilities within their areas and return to the central planners estimates of the national defense force levels and structure which they can supply at the given indices. The central planners, reflecting the preferences of the Secretary of Defense, adjust the prospective indices in a way that they believe will give preferred national defense. Conceptually the process continues until the Secretary of Defense is satisfied (supply equals demand in terms of the economic analogy). The final plan for national defense is "the plan." It is the purpose of this paper to study "the

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[1] Malinvaud, Edmond, and Bacharach, M. O. L., eds., "Decentralized Procedures For Planning" by Edmond Malinvaud, Activity Analysis in the Theory of Growth and Planning; Proceedings of a Conference held by the International Economic Association, pp. 180-181, St. Martin's Press, 1967.

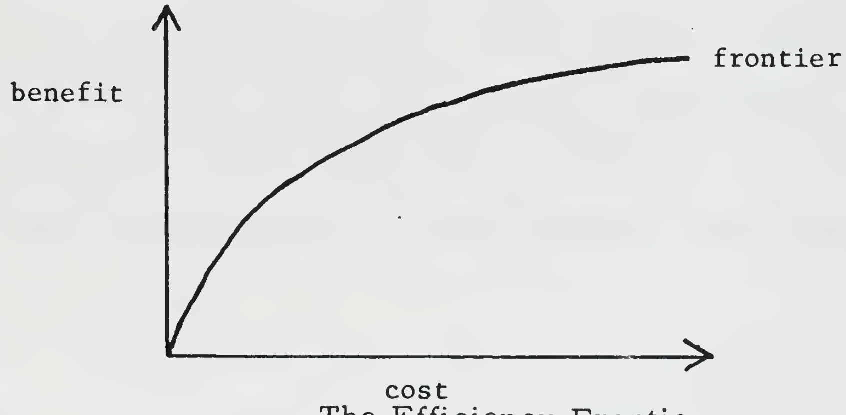
plan." "The plan" is studied without regard for the process by which it is computed in the real world.

In this paper the PPB problem is modeled so that the solution, "the plan," can be studied. The concept of efficiency as it applies to the model is discussed in the Efficiency section. The general structure of the model and the nature of the variables are presented in the section entitled Structure of the Model. Activity analysis and the mathematical structure of the submodels are explained in the following two sections. The efficiency problem is then formulated and decision rules which characterize efficient choice are derived. In the last three sections of the paper discounting of benefits over time is treated, sensitivity analysis of the model is performed, and decentralization possibilities in the model are explored.

II. EFFICIENCY

Any number of criteria can be used in choosing among alternatives. In this model, as in many economic models, alternatives will be judged using the criterion of efficiency. This is the same notion of efficiency which is part of the theory of production: production of goods is efficient if and only if no more of any good can be produced without decreasing the production of some other good.

The PPB problem can be studied in terms of benefits and costs. This approach assumes that the decision-maker's preferences about alternatives apply to a cost-benefit space. Figure 1 is a common graphical representation of the problem of choice when a single benefit and single cost are involved. The frontier and the area under it represent the set of cost-benefit combinations given by alternatives which are technologically feasible. Here technology refers



The Efficiency Frontier
Figure 1

to real world production capability. Assume that the decision-maker is not satiable with cost or benefit; that is, he always prefers less cost and more benefit. Then some alternatives dominate others by giving greater benefit at the same cost or the same benefit at less cost. Only alternatives on the frontier are undominated. These alternatives are efficient. An alternative on the frontier is efficient because no other feasible alternative can increase benefit without increasing cost or decrease cost without decreasing benefit.

A cost-benefit approach is used in this model, but the nature of the PPB problem does not generally permit the use of a single measure of benefit, nor even of a single measure of cost. In this paper the benefits are measures of effectiveness. It should be clear that the effectiveness of today's weapons can be measured in many different ways. In fact, the art of choosing measures of effectiveness has grown as systems analysis and operations research in the Department of Defense has expanded. Cost may also be measured in different ways. For example, both present discounted cost and total outlay may be pertinent to PPB decision-making in the Department of Defense. For these reasons multiple measures of effectiveness and cost are used in this model.

This does not mean that the results of Figure 1 for the single benefit-single cost case do not apply in this model. The representation of this model must, however, be

of higher dimensionality. Instead of two dimensions for one benefit and one cost, L + M dimensions are required to represent L benefits and M measures of cost. There still exists an efficiency frontier, or hypersurface, in L + M dimensions. Of course, the higher dimensional representation cannot be visualized geometrically. Hitch and McKean graphically construct in two dimensions the efficiency frontier for the two benefit-single cost case.[2] However, they must assume that the cost is constrained by a budget in order to limit the construction to two dimensions.

The cost-benefit nature of this model has still more dimensions than required by the multiple benefits and costs. An essential feature of decision-making in the PPB context is time. The planning horizon is the time period over which plans are formulated. Therefore, the planning horizon is divided into T time periods which appear explicitly in the model. The time period numbers, 1, ..., T, act as a discrete measure of time. Since benefits exist in each time period the dimensionality of the cost-benefit structure is $T \cdot L + M$. In the higher dimensional hypersurface a frontier representing efficient alternatives, efficient over time as well as within time periods, still exists. The efficient frontier represents the set from

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[2] Hitch, Charles J., and McKean, Roland N., The Economics of Defense in the Nuclear Age, pp. 379-382, Harvard University Press, 1963.

which "the plan" is chosen in the model. The choice of "the plan" for the planning period is made at the beginning of the planning period. The preferences of the decision-maker are the basis of the choice. So that all possible preferences are considered the efficient frontier is studied in this model.

III. STRUCTURE OF THE MODEL

The PPB problem could be modeled in many ways. As in economics, the phenomenon under study can be modeled in different ways, depending upon what aspects of the phenomenon are of interest. Conceptually models in economics can be as general or total as desired, but usually the more general an economic model is, the fewer theorems and conclusions it yields.

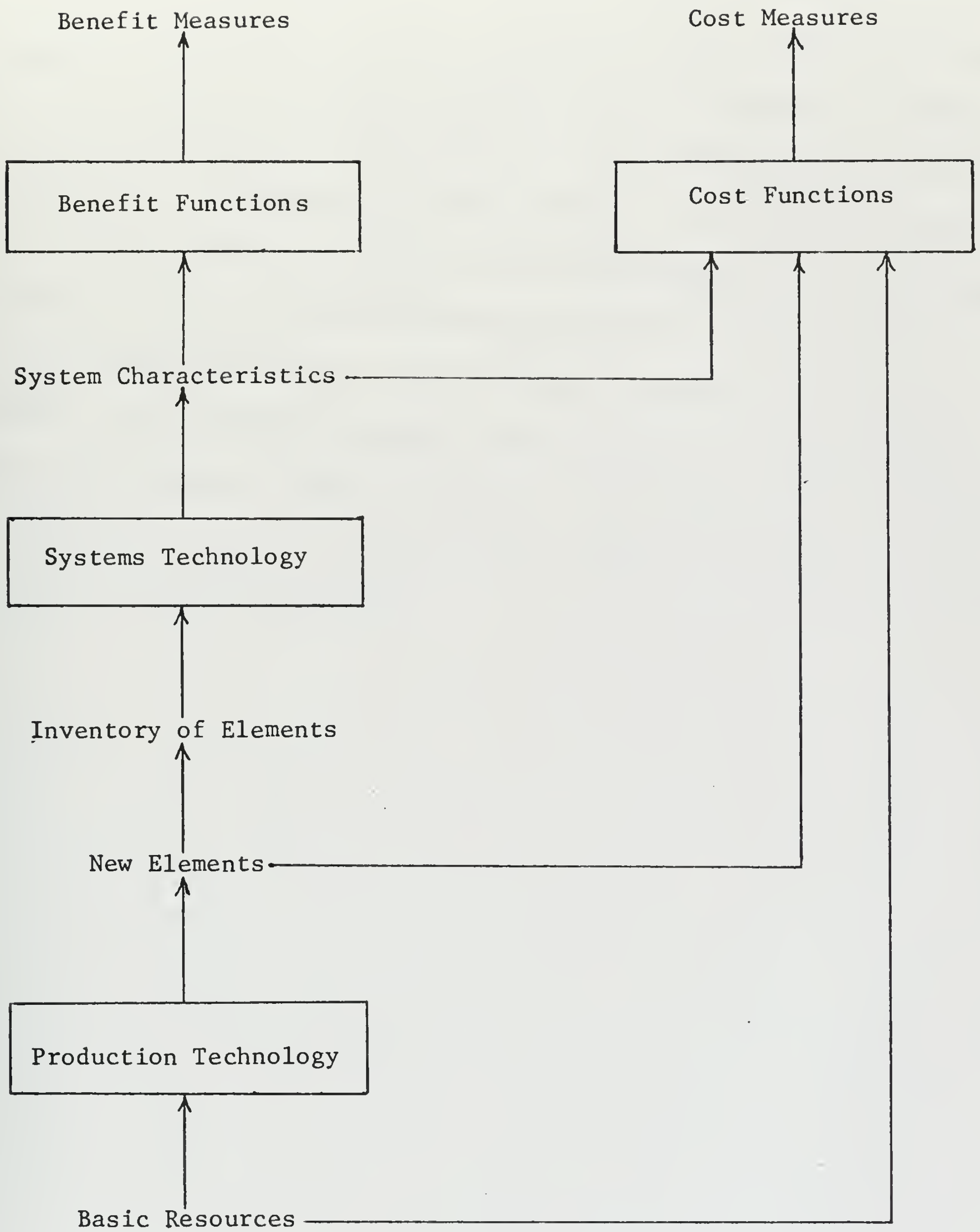
One way to model a PPB system is to break down the process by which the model's benefits and costs are generated in a time period. The process can be described in terms of basic resources, elements, and system characteristics. In a time period basic resources (e.g. man hours, raw materials) are used in the production of elements (e.g. rifles, trained servicemen), elements are combined into systems which have characteristics (e.g. a platoon's firepower, its water requirements), and system characteristics are transformed into measures of effectiveness or benefits (e.g. kill probability against a bunker). Costs are estimated by the resources used, elements produced, and characteristics existing in all time periods. Costs are an increasing function of these variables. To summarize: in this model of the PPB problem the variables are resources, elements, characteristics, benefits, and costs. The variables have been chosen so that an understanding of

their relationships to each other within the model will give some understanding of "the plan." They are variables with which it is believed a decision-maker is logically concerned.

Note that the variables in the model are not structured exactly as the program elements and programs of the PPB system used by the Department of Defense.[3] Programs are sets of effectiveness types. They are essentially classification devices, allowing costs to be linked with program effectiveness outputs rather than program inputs. An alternative specified in terms of the variables of this model can be described in terms of programs.

Figure 2 is a schematic of the model. The input-output structure of the model is obvious. The boxes connecting inputs to outputs are explained later. In the boxes the methods by which inputs are transformed into outputs are modeled. The transformation is linear in all cases except for the cost output. To explain the time aspects of the variables in the model it is helpful to distinguish between stocks and flows. Stocks exist over many time periods and are storeable, while flows have a per-time-period nature. Basic resources constitute a flow over time. Resources used in previous periods cannot be stored, so more resources are required each period for production in

[3]Hitch, Charles J., "Program Budgeting: An Appraisal," Tax Review, Vol. XXIX, No. 7, July 1968.



Schematic of the PPB Problem

Figure 2

that period. New system elements flow out of production in each period but, since they are storeable, enter the inventory of elements as a stock. Once an element is produced it lasts for the remainder of the planning period. There is no attrition of elements as time passes. Characteristics and, in turn, benefits flow each period from the inventory of elements. Cost, however, is a stock concept. All the measures of cost are some aggregation of the costs incurred over the whole planning horizon. For example, total expenditure over the planning horizon is a single number and a stock concept. All variables, except costs, have an explicit time attribute. The time period in which a variable exists is denoted by a superscript equal to the number of the time period.

IV. ACTIVITY ANALYSIS

The general structure of the model has been described in the schematic. Let a box and the box's inputs and outputs be called a submodel. Before the submodels can be presented mathematically it is necessary to describe activity analysis. Activity analysis is used in modeling the transformation of inputs into outputs in the boxes of the schematic.

As used in this model an activity is a single, linear, fixed coefficient transformation of inputs into outputs. Activity analysis as used in this paper is presented in Lancaster, although a few changes are made in that material.[4] Assume that there is a set of all outputs and a set of all inputs for each submodel. Then an activity is represented by a column vector of length equal to the total number of outputs and inputs in the two sets. For convenience the top components of the vector correspond to the various outputs and the lower components to the inputs. If an activity produces certain outputs from certain inputs then the vector will have positive numbers for the components corresponding to these outputs and inputs and zeroes elsewhere. The positive numbers are the amounts of each output produced and each input used when the activity is

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[4]Lancaster, Kelvin, <u>Mathematical Economics</u>, pp. 98-101, The Macmillan Company, New York, 1968.

operated at unit level. The unit level of an activity is chosen arbitrarily and does not change once fixed. Selecting the level at which the activity is operated, the activity level, chooses a positive scalar by which all components of the vector are multiplied. The product of the column vector and the scalar activity level is a new column vector whose components are the quantities of outputs produced and inputs used when the activity is operated at the activity level.

For a given submodel there are many activities, represented by vectors of the same length with different nonzero elements in different places. These column vectors can be placed side by side to form a matrix. The number of rows equals the total number of outputs and inputs in the submodel. The number of columns equals the number of activities. This matrix is called the technology matrix. The activity levels for the activities in the matrix can be arranged in a column vector. The column vector is of length equal to the number of activities. Its first element is the level at which the first activity (first column of the matrix) is operated. Its last element is the level at which the last activity (last column of the matrix) is operated. The product of the technology matrix and the column vector of activity levels is another column vector of length equal to the total number of outputs and inputs in the submodel. Its elements are the total quantities of each output produced and input used when all of the

activities in the matrix are operated at the specified activity levels. When the activity levels are chosen the quantities of outputs produced and inputs used in the sub-model are known.

Equation Set 1 is an activity analysis model of the production of Q outputs, $O_1, O_2, \ldots, O_Q$, from P inputs, $I_1, I_2, \ldots, I_P$, in R activities. $\Theta_1, \Theta_2, \Theta_3, \ldots, \Theta_R$ are the activity levels at which the activities are operated. The total amount of an output produced is simply the sum of the amounts of the output produced by each activity operated at its activity level: $O_q = a_{q,1}\Theta_1 + a_{q,2}\Theta_2 + \ldots + a_{q,R}\Theta_R$. The same is true of any input: $I_p = a_{Q+p,1}\Theta_1 + a_{Q+p,2}\Theta_2 + \ldots + a_{Q+p,R}\Theta_R$.

Activities have certain properties. They exhibit constant returns to scale and can be operated independently of the operation of other activities. Whereas neoclassical

$$\left[\begin{array}{ccc} a_{1,1} & \cdots & a_{1,R} \\ \vdots & & \vdots \\ a_{Q,1} & \cdots & a_{Q,R} \\ \hline a_{Q+1,1} & \cdots & a_{Q+1,R} \\ \vdots & & \vdots \\ a_{Q+P,1} & \cdots & a_{Q+P,R} \end{array}\right] \cdot \left[\begin{array}{c} \Theta_1 \\ \vdots \\ \Theta_R \end{array}\right] = \left[\begin{array}{c} O_1 \\ \vdots \\ O_Q \\ \hline I_1 \\ \vdots \\ I_P \end{array}\right]$$

Equation Set 1

production theory in economics was concerned with substitutability between primary inputs, activity analysis emphasizes complementary inputs and the joint production of outputs. However, the existence of the same output in alternative activies can allow substitution of the inputs of the alternative activities.

Many real world transformations of inputs into outputs can be modeled using activity analysis. Activity analysis can be used to model the transformation of inputs into outputs even when real world processes do not correspond in a one-to-one manner with the activities. Individual activities do not necessarily have to represent individual real world processes. Activity analysis can be used whenever the real world relations between inputs and outputs, no matter how the real world transformations are made, can be satisfactorily represented by activity analysis. If real world processes do correspond to activities, activity analysis is all the more useful. Activity analysis can be used in the same way that implicit functions are often used in economics, as a tool to keep track of inputs and outputs. It is only necessary that real world relationships between inputs and outputs be consistent with the properties of activity analysis. The most important properties of activities in this respect are: the linearity of the transformation of inputs into outputs, constant returns to scale, and the independence of activities.

V. THE SUBMODELS

The mathematical structure of the submodels can now be presented. Each submodel in each time period will have the mathematical structure of Equation Set 1 in the preceding section. The technology matrices for the submodels in a time period model the transformations of inputs into outputs which are expected to exist in that time period. The technology matrices represent the predictions which are made at the beginning of the planning period when "the plan" is chosen.

The first submodel in the schematic is the production submodel, in which basic resources are used to produce new elements. Assume that over the whole planning horizon the total number of basic resources which can ever be used as inputs to this submodel is K and the total number of new elements which can ever be produced is J. Not every basic resource may exist in every time period and it may not be possible to produce all system elements in each time period. However, K and J are the total numbers of resources and elements, each of which can exist at some time within the planning horizon. Denote the amount of the k^{th} basic resource used in period t by $x_k^t, k = 1, \ldots, K$ and the amount of the j^{th} new element produced in period t by $y_j^t, j = 1, \ldots, J$. Assume that there are γ^t activities in period t. Also assume that each activity has as

output only one new element and that each element is produced by only one activity. There is no limit set on the number of resources which may be required to produce one element. This is a restricted form of general activity analysis called input-output analysis.[5] Since one activity produces one element the number of activities, γ^t, is less than or equal to the number of elements, J. Unless all J elements are produced in a single time period $\gamma^t < J$. The unit level of each activity is that level of the activity which produces one unit of the new element. The activity analysis production submodel in a single time period is given by Equation Set 2. Here ω_g^t, $g = 1, \ldots, \gamma^t$, are the activity levels. The top part of the technology matrix contains only ones and zeroes and need not be arranged as an identity submatrix: the activities (columns) can be arranged in any order. If a basic resource cannot

$$
\left[\begin{array}{ccc}
1 & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 1 \\
\hline
p^t_{J+1,1} & \cdots & p^t_{J+1,\gamma^t} \\
\vdots & & \\
p^t_{J+K,1} & \cdots & p^t_{J+K,\gamma^t}
\end{array}\right]
\cdot
\left[\begin{array}{c}
\omega^t_1 \\
\vdots \\
\omega^t_{\gamma^t}
\end{array}\right]
=
\left[\begin{array}{c}
y^t_1 \\
\vdots \\
y^t_J \\
\hline
x^t_1 \\
\vdots \\
x^t_K
\end{array}\right]
$$

Equation Set 2

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[5] Ibid., p. 79.

be used or an element cannot be produced because of technology in a time period, then the corresponding row of the technology matrix will contain only zeroes. Equation Set 2 can be written in somewhat more concise notation:

$$\begin{bmatrix} \underline{\underline{P}}^{(y)t} \\ \hline \underline{\underline{P}}^{(x)t} \end{bmatrix} \cdot \begin{bmatrix} \underline{\omega}^{t,} \end{bmatrix} = \begin{bmatrix} \underline{y}^{t,} \\ \hline \underline{x}^{t,} \end{bmatrix}$$

The next submodel is the systems submodel, in which the system elements available in a time period are grouped into systems which have characteristics. Denote the quantity of elements of type j, $j = 1, \ldots, J$, used as input to the submodel in period t by L_j^t. Since elements last for the whole planning period L_j^t is equal to the sum of all the elements produced up to and in the time period $(L_j^t = y_j^1 + y_j^2 + \ldots + y_j^t)$. Assume that the total number of system characteristics, each of which can exist at some time within the planning horizon, is I. Denote the amounts of system characteristics in period t by z_i^t, $i = 1, \ldots, I$.

Assume that each activity in the activity analysis model of the systems submodel represents one system, one way of grouping elements together. Then the activity analysis systems submodel is given by Equation Set 3.

$$
\left[\begin{array}{ccc}
s^t_{1,1} & \cdots & s^t_{1,\beta^t} \\
\vdots & & \vdots \\
s^t_{I,1} & \cdots & s^t_{I,\beta^t} \\
\hline
s^t_{I+1,1} & \cdots & s^t_{I+1,\beta^t} \\
\vdots & & \vdots \\
s^t_{I+J,1} & \cdots & s^t_{I+J,\beta^t}
\end{array}\right] \cdot
\left[\begin{array}{c}
\zeta^t_1 \\ \cdot \\ \cdot \\ \cdot \\ \zeta^t_{\beta^t}
\end{array}\right] =
\left[\begin{array}{c}
z^t_1 \\ \vdots \\ z^t_I \\ \hline L^t_1 \\ \vdots \\ L^t_J
\end{array}\right]
$$

Equation Set 3

An activity, or system, may be a grouping of many elements and may have many characteristics, so the technology matrix has no simple form. Equation Set 3 can be written:

$$
\left[\begin{array}{c}
\underline{\underline{S}}^{(z)t} \\ \hline \underline{\underline{S}}^{(L)t}
\end{array}\right] \cdot
\left[\underline{\zeta}^{t\prime}\right] =
\left[\begin{array}{c}
\underline{z}^{t\prime} \\ \hline \underline{L}^{t\prime}
\end{array}\right]
$$

The third submodel is the benefit submodel, in which system characteristics are transformed into measures of effectiveness or benefits. Let the total number of benefits, each of which can exist at some time in the planning horizon, be L. Then denote the amount of l^{th} measure of effectiveness or benefit in time period t by E^t_l, $l = 1, \ldots, L$. Assume that each activity in the benefit submodel transforms a single system characteristic into many benefits, that one characteristic can be the input to only one activity, and that the unit level of an activity

requires one unit of the system characteristic as input. Assume that all characteristics are transformed into benefits and that there are α^t activities in time period t. Then the number of activities, α^t, in a time period is less than the number of characteristics, I, unless all characteristics exist in the time period, in which case $\alpha^t = I$. The activity analysis benefit submodel is given in Equation Set 4.

$$\left[\begin{array}{ccc} b^t_{1,1} & \cdots & b^t_{1,\alpha^t} \\ \vdots & & \vdots \\ b^t_{L,1} & \cdots & b^t_{L,\alpha^t} \\ \hline 1 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{array}\right] \cdot \begin{bmatrix} \eta^t_1 \\ \vdots \\ \eta^t_{\alpha^t} \end{bmatrix} = \left[\begin{array}{c} E^t_1 \\ \vdots \\ E^t_L \\ \hline z^t_1 \\ \vdots \\ z^t_I \end{array}\right]$$

Equation Set 4

The lower part of the technology matrix need not be arranged as an identity submatrix: the activities (columns) can be arranged in any order. The top part of the technology matrix has no simple form since a single characteristic can produce many benefits. Equation Set 4 can be written:

$$\left[\begin{array}{c} \underline{\underline{B}}^{(E)t} \\ \hline \underline{\underline{B}}^{(z)t} \end{array}\right] \cdot \left[\underline{\eta}^{t\prime}\right] = \left[\begin{array}{c} \underline{E}^{t\prime} \\ \hline \underline{z}^{t\prime} \end{array}\right]$$

In summary, the benefit side of the model, made up of the production, systems, and benefit submodels, uses activity analysis for each time period. In activities resources produce single elements, elements are grouped into systems which have characteristics, and single characteristics yield benefits. The benefit side of the model is linear.

The cost side of the model is not linear. The multiple measures of cost are a function of the resources used in all time periods (all x's), the elements produced in all time periods (all y's), and the characteristics which exist in all time periods (all z's). Denote the m^{th} cost measure by C_m, $m = 1, \ldots, M$. Then $C_m = C_m (x_1^t, \ldots, x_K^t, y_1^t, \ldots, y_J^t, z_1^t, \ldots, z_I^t, t = 1, \ldots, T)$. It is assumed that all cost measures are monotonically increasing functions of the quantities of resources, elements, and characteristics. If quantities of all resources, elements, and characteristics in all time periods except Q, the quantity of one resource, element, or characteristic in one time period, are held constant, then C_m increases at an increasing rate as Q increases. This is represented graphically in Figure 3. The assumption of increasing marginal costs is based on the idea that goods and services

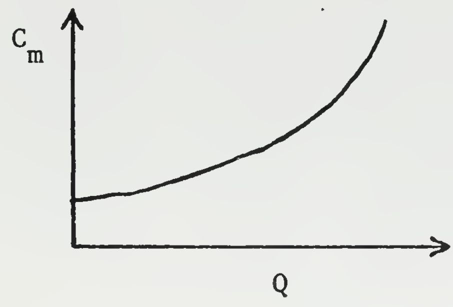

Figure 3
Cost-Quantity Relationship

must be bid away from the economy. As goods and services are purchased it becomes more costly to purchase more. The assumption conflicts with the linear cost assumptions which are made when cost estimating relations are formed by regression in the real world. However, cost estimating relations which are linear in the logs are consistent with the assumption. Costs are not broken down by categories such as research and development, investment, and operating. However, the contributions of resources and elements to the cost functions incorporate investment costs and the contributions of characteristics incorporate operating costs.

VI. THE EFFICIENCY PROBLEM

It has been stated that the efficient frontier is studied in this model. In this section a mathematical program is formulated which has as its solution an efficient alternative. By varying the parameters in the objective function of the program all efficient alternatives can be solutions.

Consider all of the submodels in all of the time periods. Using the simplified notation:

$$\underline{E}^{t\prime} = \underline{\underline{B}}^{(E)t} \cdot \underline{n}^{t\prime}$$

$$\underline{C}' = \underline{C}'\,(z_1^t, \ldots, z_I^t, y_1^t, \ldots, y_J^t, x_1^t, \ldots, x_K^t, t = 1, \ldots, T)$$

$$\underline{z}^{t\prime} = \underline{\underline{B}}^{(z)t} \cdot \underline{n}^{t\prime} = \underline{\underline{S}}^{(z)t} \cdot \underline{\zeta}^{t\prime}$$

$$\underline{L}^{t\prime} = \underline{\underline{S}}^{(L)t} \cdot \underline{\zeta}^{t\prime} = \sum_{i=1}^{t} \underline{\underline{P}}^{(y)t} \cdot \underline{\omega}^{i\prime}$$

$$\underline{y}^{t\prime} = \underline{\underline{P}}^{(y)t} \cdot \underline{\omega}^{t\prime}$$

$$\underline{x}^{t\prime} = \underline{\underline{P}}^{(x)t} \cdot \underline{\omega}^{t\prime}$$

Substituting for the z's, L's, y's, and x's the above equations become:

$$\underline{E}^{t\prime} = \underline{\underline{B}}^{(E)t} \cdot \underline{n}^{t\prime} \qquad t = 1, \ldots, T$$

$$\underline{C}' = \underline{C}'\,\Big(\sum_{b=1}^{\beta^t} s_{1,b}^t\, \zeta_b^t, \ldots, \sum_{b=1}^{\beta^t} s_{I,b}^t\, \zeta_b^t, \sum_{g=1}^{\gamma^t} p_{1,g}^t\, \omega_g^t, \ldots, \sum_{g=1}^{\gamma^t} p_{J,g}^t\, \omega_g^t, \sum_{g=1}^{\gamma^t} p_{J+1,g}^t\, \omega_g^t, \ldots, \sum_{g=1}^{\gamma^t} p_{J+K,g}^t\, \omega_g^t, t = 1, \ldots, T\Big)$$

$$\underline{\underline{B}}^{(z)t} \cdot \underline{n}^{t\prime} = \underline{\underline{S}}^{(z)t} \cdot \underline{\zeta}^{t\prime} \qquad t = 1, \ldots, T$$

$$\underline{\underline{S}}^{(L)t} \cdot \underline{\zeta}^{t\prime} = \sum_{i=1}^{t} \underline{\underline{P}}^{(y)t} \cdot \underline{\omega}^{i\prime} \qquad t = 1, \ldots, T$$

The model has been constructed in such a way that these relations are true. The relations will be constraints in the mathematical program.

Recall the efficient frontier in the single benefit-single cost choice problem. When the frontier has appropriate shape a point on the frontier can be characterized as the point of tangency of the frontier and some straight line. See Figure 4. Consider the general straight line $c = aE - bC$. The slope of the line is b/a and its E axis intercept is c. For any point on the efficiency frontier, let $b = \Phi E - \Psi C$ be the line which is tangent to the efficient frontier at the point. Then the point on the frontier can be viewed as the alternative from the feasible set which maximizes $\Phi E - \Psi C$. In terms of Figure 4 this means choosing the point from the feasible set which lies on the line of slope Ψ/Φ which has the highest E axis intercept. The intercept increases as the line moves up and to the left. The point from the feasible set which lies on the highest and farthest left line of slope Ψ/Φ is the point on the frontier. It is also true that for any $\Phi, \Psi > 0$ the alternative from the feasible set which maximizes $\Phi E - \Psi C$ is efficient. The point from

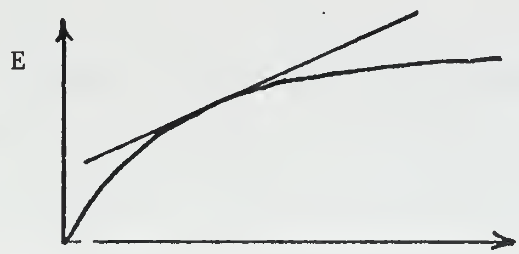

Figure 4. Tangent to the Efficiency Frontier

the feasible set which lies on the highest and farthest left line of any slope between zero and infinity is on the efficient frontier. By varying Φ and Ψ all efficient alternatives can be characterized: the efficient frontier can be "swept out."

This reasoning can be extended to higher dimensional choice problems. For example, any efficient alternative in a two benefit-single cost choice problem can be viewed as that alternative from the feasible set which maximizes $\Phi_1E_1 + \Phi_2E_2 - \Psi C$, for some Φ_1 , Φ_2 , and Ψ . The efficient alternative is the point of tangency of a plane $P = \Phi_1E_1 + \Phi_2E_2 - \Psi C$ and the feasible set in three dimensions.

When this reasoning is applied to the $T \cdot L + M$ dimensional choice problem of the model any efficient alternative can be characterized as the alternative from the feasible set which maximizes $\Phi_1^1E_1^1 + \ldots + \Phi_L^1E_L^1 + \ldots + \Phi_1^TE_1^T + \ldots + \Phi_L^TE_L^T - \Psi_1C_1 - \ldots - \Psi_MC_M$, for some Φ's and Ψ's. The alternative is the point of tangency of a hyperplane $H = \Phi_1^1E_1^1 + \ldots + \Phi_L^1E_L^1 + \ldots + \Phi_1^TE_1^T + \ldots + \Phi_L^TE_L^T - \Psi_1C_1 - \ldots - \Psi_MC_M$ and the feasible set in $T \cdot L + M$ dimensions. This maximum is, however, one possible vector maximum of E_l^t, $l = 1, \ldots, L$, $t = 1, \ldots, T$, C_m, $m = 1, \ldots, M$. The vector maximum of a vector of variables is the maximum of their weighted sum, where the weightings are arbitrary. If the weightings used in the vector maximization are Φ_l^t, $l = 1, \ldots, L$, $t = 1, \ldots, T$, $-\Psi_m$, $m = 1, \ldots, M$, then the vector

maximum is the same as the above maximum. Therefore, any efficient alternative is the solution to the mathematical program entitled The Efficiency Problem, for some set of weightings of the E's and C's.

The Efficiency Problem

$$\text{"Max"} \quad \begin{bmatrix} E_1^1 \\ \vdots \\ E_L^T \\ \hline C_1 \\ \vdots \\ C_M \end{bmatrix}$$

subject to:

$$\underline{E}^{t\prime} = \underline{\underline{B}}^{(E)t} \cdot \underline{\eta}^{t\prime} \qquad t = 1, \ldots, T$$

$$\underline{C}' = \underline{C}'\Big(\sum_{b=1}^{\beta^t} s_{1,b}^t \zeta_b^t, \ldots, \sum_{b=1}^{\beta^t} s_{I,b}^t \zeta_b^t, \sum_{g=1}^{\gamma^t} p_{1,g}^t \omega_g^t, \ldots, \sum_{g=1}^{\gamma^t} p_{J,g}^t \omega_g^t, \sum_{g=1}^{\gamma^t} p_{J+1,g}^t \omega_g^t, \ldots, \sum_{g=1}^{\gamma^t} p_{J+K,g}^t \omega_g^t, \ t = 1, \ldots, T\Big)$$

$$\underline{\underline{B}}^{(z)t} \cdot \underline{\eta}^{t\prime} = \underline{\underline{S}}^{(z)t} \cdot \underline{\zeta}^{t\prime} \qquad t = 1, \ldots, T$$

$$\underline{\underline{S}}^{(L)t} \cdot \underline{\zeta}^{t\prime} = \sum_{i=1}^{t} \underline{\underline{P}}^{(y)t} \cdot \underline{\omega}^{t\prime} \qquad t = 1, \ldots, T$$

$$E_l^t, C_m \geq 0 \qquad l = 1, \ldots, L \quad t = 1, \ldots, T \quad m = 1, \ldots, M$$

$$\underline{\eta}^{t\prime}, \underline{\zeta}^{t\prime}, \underline{\omega}^{t\prime} \geq \underline{0}' \qquad t = 1, \ldots, T$$

As in the single benefit-single cost case it is also true that the optimal solution to the efficiency problem when the weightings are any Φ's and $-\Psi$'s (Φ, $\Psi > 0$)is an efficient alternative. By varying the Φ's and Ψ's all efficient alternatives become solutions to the efficiency problem: the efficient frontier is "swept out." If the decision-maker specifies the Φ's and Ψ's, if he specifies how he weighs benefits over time and costs, then

"the plan" is the optimal solution to the efficiency problem.

Note that some of the original variables of the model do not appear in the efficiency problem. The activity levels, $\underline{\eta}^t$, $\underline{\zeta}^t$, $\underline{\omega}^t$, $t = 1, \ldots, T$, are the choice variables in the problem. The z's, y's, and x's do not appear in the efficiency problem, but they are easily computed from the activity levels by multiplying the technology matrices by the activity levels.

The feasible set in the efficiency problem has "facets" due to its activity analysis basis. For example, in the two benefit-single cost case the feasible set may be represented geometrically as in Figure 5. The four edges of the solid feasible set represent four benefit and cost producing activities; the three curved surfaces represent combinations of activity uses. An efficient alternative can always be described as the point of tangency of some hyperplane and the feasible set because of the shape of the feasible set. Everything in the constraints is linear except the cost functions, whose shapes have been described. However, the hyperplane will not be unique for alternatives on a facet

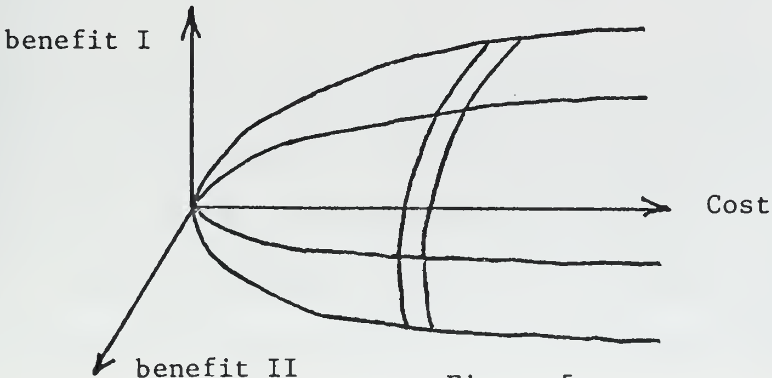

Figure 5
Three Dimensional Efficiency Frontier

of the feasible set because many hyperplanes can have the same point of tangency. As in the single benefit-single cost choice problem where Ψ/Φ gives the slope of the tangent line, the ratios of the Φ 's and Ψ 's to each other give the slopes of the hyperplane. Since only the ratios are important the Φ 's and Ψ 's can be scaled arbitrarily. This is another way of saying that any single measure of effectiveness or cost can be chosen as numéraire. When a numéraire is chosen the Φ 's and Ψ 's are divided by the weighting of the numéraire measure of effectiveness or cost. This does not affect the slopes of the hyperplane. The only change of interest is that the weighting of the numéraire becomes one.

The Lagrangian of the efficiency problem is:

$$
\begin{aligned}
L = {} & \sum_{t=1}^{T}\sum_{l=1}^{L}\Phi_l^t E_l^t - \sum_{m=1}^{M}\Psi_m C_m - \sum_{t=1}^{T}\sum_{l=1}^{L}\lambda_l^{(E)t}\left(E_l^t - \sum_{a=1}^{\alpha^t} b_{l,a}^t \eta_a^t\right) \\
& - \sum_{m=1}^{M}\lambda_m^{(C)}\Big(C_m - C_m\Big(\sum_{b=1}^{\beta^t} s_{1,b}^t \zeta_b^t, \ldots, \sum_{b=1}^{\beta^t} s_{J,b}^t \zeta_b^t, \sum_{g=1}^{\gamma^t} p_{1,g}^t \omega_g^t, \\
& \qquad \ldots, \sum_{g=1}^{\gamma^t} p_{J,g}^t \omega_g^t, \sum_{g=1}^{\gamma^t} p_{J+1,g}^t \omega_g^t, \ldots, \sum_{g=1}^{\gamma^t} p_{J+K,g}^t \omega_g^t, t = 1, \ldots, T\Big)\Big) \\
& - \sum_{t=1}^{T}\sum_{i=1}^{I}\lambda_i^{(z)t}\left(\sum_{a=1}^{\alpha^t} b_{L+i,a}^t \eta_a^t - \sum_{b=1}^{\beta^t} s_{i,b}^t \zeta_b^t\right) \\
& - \sum_{t=1}^{T}\sum_{j=1}^{J}\lambda_j^{(L)t}\left(\sum_{b=1}^{\beta^t} s_{I+j,b}^t \zeta_b^t \quad \sum_{i=1}^{t}\sum_{g=1}^{\gamma^t} p_{j,g}^i \omega_g^i\right)
\end{aligned}
$$

The Lagrangian may not be useful in computing a solution, but the Kuhn-Tucker conditions are useful in characterizing a solution to the efficiency problem.

The Kuhn-Tucker first-order conditions are necessary conditions for a maximum of the Lagrangian. If

$\hat{E}_1^t$, $l = 1, \ldots,$, $\hat{C}_m$, $m = 1, \ldots, M$, $\hat{\eta}_a^t$, $a = 1, \ldots, \alpha^t$, $\hat{\zeta}_b^t$, $b = 1, \ldots, \beta^t$, $\hat{\omega}_g^t$, $g = 1, \ldots, \gamma^t$, $t = 1, \ldots, T$, maximize the Lagrangian, then the following relations, which involve partial derivatives of the Lagrangian with respect to the choice variables and the Lagrange multipliers, must hold:

1) $\Phi_l^t - \hat{\lambda}_l^{(E)t} \leqq 0 \quad \hat{E}_l^t \geqq 0 \qquad \hat{E}_l^t\,(\Phi_l^t - \hat{\lambda}_l^{(E)t}) = 0 \qquad l = 1, \ldots, L;\ t = 1, \ldots, T$

2) $-\Psi_m - \hat{\lambda}_m^{(C)} \leqq 0 \quad \hat{C}_m \geqq 0 \qquad \hat{C}_m\,(-\Psi_m - \hat{\lambda}_m^{(C)}) = 0 \qquad m = 1, \ldots, M$

3) $\sum_{l=1}^{L} \hat{\lambda}_l^{(E)t} b_{l,a}^t - \sum_{i=1}^{I} \hat{\lambda}_i^{(z)t} b_{L+i,a}^t \leqq 0 \quad \hat{\eta}_a^t \geqq 0$

$\hat{\eta}_a^t$ (left side of above inequality) $= 0 \qquad a = 1, \ldots, \alpha^t;\ t = 1, \ldots, T$

4) $\sum_{m=1}^{M} \hat{\lambda}_m^{(C)} \sum_{i=1}^{I} \frac{\partial \hat{C}_m}{\partial \hat{z}_i^t} s_{i,b}^t + \sum_{i=1}^{I} \hat{\lambda}_i^{(z)t} s_{i,b}^t - \sum_{j=1}^{J} \hat{\lambda}_j^{(L)t} s_{I+j,b}^t \leqq 0 \quad \hat{\zeta}_b^t \geqq 0$

$\hat{\zeta}_b^t$ (left side of above inequality) $= 0 \qquad b = 1, \ldots, \beta^t;\ t = 1, \ldots, T$

5) $\sum_{m=1}^{M} \hat{\lambda}_m^{(C)} \left(\sum_{j=1}^{J} \frac{\partial \hat{C}_m}{\partial \hat{y}_j^t} p_{j,g}^t + \sum_{k=1}^{K} \frac{\partial \hat{C}_m}{\partial \hat{x}_k^t} p_{J+k,g}^t \right) + \sum_{i=t}^{T} \sum_{j=1}^{J} \hat{\lambda}_j^{(L)i} p_{j,g}^t \leqq 0$

$\hat{\omega}_g^t \geqq 0 \qquad \hat{\omega}_g^t$ (left side of above inequality) $= 0 \qquad g = 1, \ldots, \gamma^t;\ t = 1, \ldots, T$

6) $-\hat{E}_l^t + \sum_{a=1}^{\alpha^t} b_{l,a}^t \hat{\eta}_a^t = 0 \qquad \hat{\lambda}_l^{(E)t}$ unrestricted in sign $\qquad l = 1, \ldots, L;\ t = 1, \ldots, T$

7) $-\hat{C}_m + \hat{C}_m$ (domain of m^{th} cost function at optimum) $= 0$

$\hat{\lambda}_m^{(C)}$ unrestricted in sign $\qquad m = 1, \ldots, M$

8) $-\sum_{a=1}^{\alpha^t} b_{i,a}^t \hat{\eta}_a^t + \sum_{b=1}^{\beta^t} s_{i,b}^t \hat{\zeta}_b^t = 0 \qquad \hat{\lambda}_i^{(z)t}$ unrestricted in sign

$i = 1, \ldots, I \qquad t = 1, \ldots, T$

9) $-\sum_{b=1}^{\beta^t} s_{j,b}^t \hat{\zeta}_b^t + \sum_{i=1}^{I} \sum_{g=1}^{\gamma^t} p_{j,g}^i \hat{\omega}_g^i = 0 \qquad \hat{\lambda}_j^{(L)t}$ unrestricted in sign

$j = 1, \ldots, J \qquad t = 1, \ldots, T$

The number of variables, $T(L + \alpha + \beta + \gamma) + T(L + I + J) + 2M$, equals the number of relations. The shape of the efficiency problem's feasible set, linear except for the cost

functions, and its linear objective function show that the Lagrangian has the concavity-convexity properties which make the Kuhn-Tucker conditions sufficient as well as necessary. This means that any set of benefits, costs, activity levels, and Lagrange multipliers which satisfy the above relations is a solution to the efficiency problem and defines an efficient alternative.

The Lagrange multipliers are weightings which distribute, or impute, the value of the objective function to the constraints. All of the constraints can be written in the form: left hand side minus right hand side equals zero. For example:

$$\hat{E}_1^t - \sum_{a=1}^{\alpha^t} b_{1,a}^t \hat{\eta}_a^t = 0$$

The Lagrange multiplier for a constraint is approximately the increase in the objective function which would be possible if the zero for that constraint were to become a one. Since each constraint is in terms of a single variable, this is the same as the increase in the objective function if one more unit of the constraint variable were available. Therefore, the Lagrange multiplier for a constraint is the marginal value to the objective function of the constraint variable. The constraint variables are E 's, C 's, z 's, and L 's. The Lagrange multipliers are marginal, or imputed, values of the E 's, C 's, z 's, and L 's. For the example $\hat{\lambda}_1^{(E)t}$ is the imputed value of $\hat{E}_1^t$.

The Kuhn-Tucker conditions can be interpreted using the meanings of the Lagrange multipliers. Since the conditions are necessary for an efficient alternative, the interpreted conditions characterize any efficient alternative. Since the conditions are sufficient, the interpreted conditions can be viewed as decision rules which guarantee efficiency. If the decision rules are true for an alternative, then the alternative is efficient. Consider the third condition:

$$\sum_{l=1}^{L} \hat{\lambda}_l^{(E)t} b_{l,a}^t \leqq \sum_{i=1}^{I} \hat{\lambda}_i^{(z)t} b_{L+i,a}^t$$

The third condition says that for unit level of operation of any activity a in the benefit submodel, $a = 1, \ldots, \alpha^t$, the weighted output of the activity (weighted by imputed value) is less than or equal to the weighted input (weighted by imputed value). If output of an activity is called product and its input is called cost (not necessarily related to the costs in the objective function), then weighted product of an activity is less than or equal to its imputed cost. Recall that an activity in the benefit sub-model converts a single system characteristic into measures of effectiveness. Therefore, the weighted product of any system characteristic is less than or equal to its imputed cost. If the activity is operated at a positive level, $\hat{\eta}_g^t > 0$, then the weighted product of the activity equals the imputed cost. This means that the weighted product of any system characteristic which exists in a time period is equal to its weighted cost. Take the ratio of the third

conditions for any two activities, a_1 and a_2, in the same or different time periods, t_1 and t_2, which are operated at positive level:

$$\frac{\sum_{l=1}^{L} \hat{\lambda}_l^{(E)t_1} b_{l,a_1}^{t_1}}{\sum_{l=1}^{L} \hat{\lambda}_l^{(E)t_1} b_{l,a_2}^{t_2}} = \frac{\sum_{i=1}^{I} \hat{\lambda}_i^{(z)t_1} b_{L+i,a_1}^{t_1}}{\sum_{i=1}^{I} \hat{\lambda}_i^{(z)t_2} b_{L+i,a_2}^{t_2}}$$

The ratio of weighted products in measures of effectiveness of two system characteristics existing in the same or different time periods equals the ratio of their imputed costs.

Consider the fourth condition:

$$\sum_{i=1}^{I} \hat{\lambda}_i^{(z)t} s_{i,b}^{t} \leqq \sum_{j=1}^{J} \hat{\lambda}_j^{(L)t} s_{I+j,b}^{t} - \sum_{m=1}^{M} \hat{\lambda}_m^{(C)} \sum_{i=1}^{I} \frac{\partial \hat{C}_m}{\partial \hat{z}_i^t} s_{i,b}^{t}$$

It says that the weighted product of any activity b, $b = 1, \ldots, \beta^t$, in the systems submodel at unit level of operation is less than or equal to its imputed cost. Here the imputed cost is composed of two factors: weighted inputs (first term) and weighted contribution to the cost functions of the objective function (second term). The second term is positive because the $\hat{\lambda}_m$'s$(=-\Psi_m$'s) are negative. Recall that an activity in the systems submodel represents a system. This means that the weighted product in system characteristics of any possible system in any time period is less than or equal to its imputed cost. If the activity is operated at positive level, $\hat{\zeta}_b^t > 0$, and the system exists in the time period, then the weighted product in system characteristics of the system equals its imputed cost. If two systems, b_1 and b_2, exist in the same or

different time periods, t_1 and t_2, the ratio of their weighted products in system characteristics equals the ratio of their imputed costs:

$$\frac{\sum_{i=1}^{I} \hat{\lambda}_i^{(z)t_1} s_{i,b_1}^{t_1}}{\sum_{i=1}^{I} \hat{\lambda}_i^{(z)t_2} s_{i,b_2}^{t_2}} = \frac{\sum_{j=1}^{J} \hat{\lambda}_j^{(L)t_1} s_{I+j,b_1}^{t_1} - \sum_{m=1}^{M} \hat{\lambda}_m^{(C)} \sum_{i=1}^{I} \frac{\partial \hat{C}_m}{\partial \hat{z}_i^{t_1}} s_{i,b_1}^{t_1}}{\sum_{j=1}^{J} \hat{\lambda}_j^{(L)t_2} s_{I+j,b_2}^{t_2} \quad \sum_{m=1}^{M} \hat{\lambda}_m^{(C)} \sum_{i=1}^{I} \frac{\partial \hat{C}_m}{\partial \hat{z}_i^{t_2}} s_{i,b_2}^{t_2}}$$

Consider the fifth condition:

$$\sum_{i=t}^{T} \sum_{j=1}^{J} \hat{\lambda}_j^{(L)i} p_{j,g}^{t} \leq - \sum_{m=1}^{M} \hat{\lambda}_m^{(C)} \left(\sum_{j=1}^{J} \frac{\partial \hat{C}_m}{\partial \hat{y}_j^{t}} p_{j,g}^{t} + \sum_{k=1}^{K} \frac{\partial \hat{C}_m}{\partial \hat{x}_k^{t}} p_{J+k,g}^{t} \right)$$

It says that the weighted product of any activity g, $g = 1, \ldots, \gamma^t$ in the production submodel is less than or equal to its imputed cost. Here both cost terms are contributions to the cost functions of the objective function, again positive because the $\hat{\lambda}_m^{(C)}$'s are negative. Note that the activity's output is weighted by the present and all future Lagrange multipliers for elements. This reflects the assumption that elements, once produced, last and contribute to the objective function for the duration of the planning period.

This means that the product of an activity in an early time period is weighted more heavily than the product of an activity in a later time period, since the earlier weighting is the sum of the later weighting and all intervening Lagrange multipliers for elements.

Recall that an activity in the production submodel has a single element as output. Therefore, the value of any possible element, weighted in its present and future time

periods, is less an or equal to its imputed cost. If the activity is operated at positive level, $\hat{\omega}^t_g > 0$, and the element is actually produced, its worth in present and future time periods is equal to its imputed cost. If two elements, g_1 and g_2 , are actually produced in the same or different time periods, t_1 and t_2, then the ratio of their total weighted values is equal to the ratio of their imputed costs:

$$\frac{\sum_{i=t_1}^{T}\sum_{j=1}^{J}\hat{\lambda}_j^{(L)i}\,p_{j,g_1}^{t_1}}{\sum_{i=t_2}^{T}\sum_{j=1}^{J}\hat{\lambda}_j^{(L)i}\,p_{j,g_2}^{t_2}} = \frac{-\sum_{m=1}^{M}\hat{\lambda}_m^{(C)}\left(\sum_{j=1}^{J}\frac{\partial\hat{C}_m}{\partial\hat{y}_j^{t_1}}\,p_{j,g_1}^{t_1} + \sum_{k=1}^{K}\frac{\partial\hat{C}_m}{\partial\hat{x}_k^{t_1}}\,p_{J+k,g_1}^{t_1}\right)}{-\sum_{m=1}^{M}\hat{\lambda}_m^{(C)}\left(\sum_{j=1}^{J}\frac{\partial\hat{C}_m}{\partial\hat{y}_j^{t_2}}\,p_{j,g_2}^{t_2} + \sum_{k=1}^{K}\frac{\partial\hat{C}_m}{\partial\hat{x}_k^{t_2}}\,p_{J+k,g_2}^{t2}\right)}$$

VII. DISCOUNTING

It has been stated that, if the decision-maker specifies the Φ's and Ψ's of the objective function of the efficiency problem, then the optimal solution of the efficiency problem is "the plan." The Φ's and Ψ's reflect the way in which the decision-maker trades off benefits over time and costs. When the Φ's and Ψ's are specified the objective function of the efficiency problem can be viewed as the present value to the decision-maker of benefits and costs over the planning horizon. "The plan" is chosen at the beginning of the planning period so that the present value of the benefits and costs consumed over the planning horizon is maximized. The concept of present value of future consumption involves discounting. As is usual in economics discounting reflects the preferences of the decision-maker over time.

The objective function is:

$$\sum_{t=1}^{T} \sum_{l=1}^{L} \Phi_l^t E_l^t - \sum_{m=1}^{M} \Psi_m C_m$$

Since the benefits have a time attribute the decision-maker's discounting of benefits can be studied conventionally. The decision-maker's preferences over time for benefits are reflected in the Φ's. Consider the weightings of a single benefit over time, Φ_1^t, $t = 1, \ldots, T$. Define d_i's so that the following relation is true for all t:

$$\Phi_1^t = \frac{\Phi_1^1}{\prod_{i=1}^{t} (1 + d_i)}$$

Then d_i can be called the decision-maker's "marginal rate of psychological discount" for the 1^{th} benefit in time period i. It is also possible to view $(1 + d_i)$ as the "marginal discount factor."[6] Note that Φ_1^t, $t = 1, \ldots, T$ could be divided by Φ_1^1 and the meaning of d_i would remain unchanged. For discounting purposes Φ_1^1 acts as numéraire.

The change in 1^{th} benefit in period t, ΔE_1^t, which just compensates the decision-maker for a unit loss in the 1^{th} benefit in period one is:

$$\Delta E_1^t = \prod_{i=1}^{t} (1 + d_i)$$

If the decision-maker always prefers benefit in an early time period to the same benefit in later time periods, he has positive time preferences in all time periods for that benefit. This means that $d_i > 0$ for $i = 1, \ldots, T$ for that benefit. If he always prefers benefit in a late time period to the same benefit in early time periods, he has negative time preferences in all time periods for that benefit. This means that $d_i < 0$, $i = 1, \ldots, T$, for that benefit.

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[6] Kuenne, Robert E., The Theory of General Economic Equilibrium, p. 231, Princeton University Press, 1963.

The decision-maker can discount benefits in any way that he wishes. It is often assumed in economics that consumers have positive time preferences for commodities because in consuming they forego investment which offers a positive rate of return. In this model, however, the decision-maker may have negative time preference. He may prefer to have benefits later rather than sooner. In economics rates of psychological discount are sometimes assumed to be constant over time. There appears to be no basis for such an assumption in this model.

A decision-maker can have neutral time preferences for the 1^{th} benefit in all time periods. This means that $d_i = 0, i = 1, \ldots, T$ for the 1^{th} benefit and that $\phi_1^1 = \phi_1^2 = \ldots = \phi_1^T$. Neutral time preferences for all benefits will result in greater total production of benefits in all time periods than will occur if any time preferences are positive or negative. Adding up benefits over time periods assumes that the time periods are weighted equally. If the decision-maker has neutral time preferences, then individual benefits are weighted equally in all time periods in the objective function. When the objective function is maximized the sum of each benefit over time is maximized. If a decision-maker has positive time preferences, he accepts less total benefits in all time periods so that he can consume more benefits earlier.

Note that even with neutral time preferences for all measures of effectiveness, more elements will be produced

in the production submodel in early time periods than in later periods. This is because elements produced early are grouped into systems whose characteristics yield benefits, measures of effectiveness, over the whole remaining planning horizon. This is reflected in the weighting of elements by the sum of their Lagrange multipliers for the remaining planning period. Measures of effectiveness in different time periods are balanced by producing elements early, so that the elements yield balanced measures of effectiveness for the remainder of the planning period.

VIII. SENSITIVITY ANALYSIS

In this section the effect of changes in the parameters of the efficiency problem on the optimal solution to that problem is studied. This study can be called sensitivity analysis or comparative statics. The comparative statics label emphasizes that the analysis is static. The optimal solution to two mathematical programs, the original efficiency problem and a new efficiency problem in which one or more of the parameters of the original have been changed, are compared. Nothing is said about the dynamics of the change from one optimal solution to another.

The efficiency problem is a nonlinear programming problem. The nonlinearity is due to the cost functions. The study of the effect of variations in parameters, or sensitivity analysis, of the model cannot depend on simple theorems about parameter variations, as would be the case if the problem were linear. However, the fact that an optimal solution to the efficiency problem is a saddle point of the Lagrangian can be used as a basis for sensitivity analysis of the model.

Let $L(\underline{x}, \underline{\lambda})$ be the original Lagrangian of the efficiency problem, where $\underline{x} = E_l^t, C_m, \eta_a^t, \zeta_b^t, \omega_g^t$, $l = 1, \ldots, L$, $m = 1, \ldots, M$, $a = 1, \ldots, \alpha^t$, $b = 1, \ldots, \beta^t$, $g = 1, \ldots, \gamma^t$, $t = 1, \ldots, T$ and $\underline{\lambda} = \lambda_l^{(E)t}, \lambda_m^{(C)}, \lambda_i^{(z)t}, \lambda_j^{(L)t}$, $l = 1, \ldots, L$, $m = 1, \ldots, M$, $i = 1, \ldots, I$, $j = 1, \ldots, J$, $t = 1, \ldots, T$.

Let $L'(\underline{x}, \underline{\lambda})$ be the Lagrangian of the efficiency problem in which a parameter has been changed. Let $\underline{\hat{x}}, \underline{\hat{\lambda}}$ be an optimal solution to the original efficiency problem and let $\underline{\bar{x}}, \underline{\bar{\lambda}}$ be an optimal solution to the efficiency problem with a parameter change. Then $L(\underline{\hat{x}}, \underline{\hat{\lambda}})$ and $L'(\underline{\bar{x}}, \underline{\bar{\lambda}})$ are saddle points of L and L', respectively. The definition of a saddle point gives:

$$L(\underline{\hat{x}}, \underline{\bar{\lambda}}) \geqq L(\underline{\hat{x}}, \underline{\hat{\lambda}}) \geqq L(\underline{\bar{x}}, \underline{\hat{\lambda}})$$

$$L'(\underline{\bar{x}}, \underline{\hat{\lambda}}) \geqq L'(\underline{\bar{x}}, \underline{\bar{\lambda}}) \geqq L'(\underline{\hat{x}}, \underline{\bar{\lambda}})$$ [7]

These inequalities imply:

$$L'(\underline{\bar{x}}, \underline{\hat{\lambda}}) - L(\underline{\bar{x}}, \underline{\hat{\lambda}}) \geqq L'(\underline{\hat{x}}, \underline{\bar{\lambda}}) - L(\underline{\hat{x}}, \underline{\bar{\lambda}})$$

Recall that any parameter in either Lagrangian is multiplied by at most a single variable and a single Lagrange multiplier. Any parameters which do not change are eliminated in the above inequality by the subtraction of one Lagrangian from the other, because the Lagrangians have the same variables and Lagrange multipliers in their domains. This means that if a single parameter changes:

$$\Delta\text{parameter } (\bar{x}, \hat{\lambda} - \hat{x}, \bar{\lambda}) \geqq 0$$

where $\hat{x}, \hat{\lambda}$ and $\bar{x}, \bar{\lambda}$ are multiplied by the changing parameter in L and L', respectively. When two parameters of the efficiency problem are changed:

$$\Delta\text{parameter}_1 (\bar{x}_1, \hat{\lambda}_1 - \hat{x}_1, \bar{\lambda}_1) + \Delta\text{parameter}_2 (\bar{x}_2, \hat{\lambda}_2 - \hat{x}_2, \bar{\lambda}_2) = 0$$

where x_1's, λ_1's and x_2's, λ_2's are multiplied by the first and second changing parameters, respectively.

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[7] Hadley, G., Nonlinear and Dynamic Programming, p. 75, Addison-Wesley Publishing Company, Inc., 1964.

These inequalities can be used to evaluate the effect of a change in one or two parameters of the efficiency problem. It will be seen that the effect is precisely specified by the inequalities when the changing parameter is multiplied only by a variable and not by a Lagrange multiplier in the Lagrangian.

Changes in the weightings of benefits, the Φ 's, are first considered. Then changes in the weightings of the cost measures, the ψ 's, are studied. Finally changes in components of the submodel technology matrices are treated.

The effects of changes in the Φ 's and Ψ 's are of special interest because it is changes in these parameters which "sweep out" the efficient frontier. Consider an increase in one of the Φ 's. Let Φ_1^t be increased by $\Delta\Phi_1^t$ to $\Phi_1^{t\,\prime}$. Since Φ_1^t and $\Phi_1^{t\,\prime}$ are multiplied only by a variable, E_1^t, in L and L', respectively:

$$\Delta\Phi_1^t \, (\bar{E}_1^t - \hat{E}_1^t) \geqq 0$$

This means that increasing the weighting of a benefit in the objective function of the efficiency problem can only increase or leave unchanged the optimal amount of the benefit produced. Consider an increase in two Φ 's, $\Phi_{1_1}^{t1}$ and $\Phi_{1_2}^{t2}$. The inequality is:

$$\Delta\Phi_{1_1}^{t1} \, (\bar{E}_{1_1}^{t1} - \hat{E}_{1_1}^{t1}) + \Delta\Phi_{1_2}^{t2} \, (\bar{E}_{1_2}^{t2} - \hat{E}_{1_2}^{t2}) \geqq 0 \quad .$$

This relation allows no statement about the direction of change of the individual optimal benefits. Rather it allows complicated statements about how the optimal benefits vary together. For example, the inequality is equivalent to:

$$- \frac{\Delta\Phi^{t1}_{1_1}}{\Delta\Phi^{t2}_{1_2}} \leqq \frac{(\bar{E}^{t2}_{12} - \hat{E}^{t2}_{1_2})}{(\bar{E}^{t1}_{1_1} - \hat{E}^{t1}_{11})}$$

This version of the inequality has no simple interpretation. Similar conclusions follow from changing two Φ 's in any way, not just increasing them.

The effect of a change in a Ψ is evaluated in the same way as is the effect of a change in a Φ. Increasing a Ψ parameter makes the $-\Psi$ weighting of a cost measure more negative. Since the $-\Psi$ weightings are negative, the direction of movement of the cost measure which is weighted by a $-\Psi$ is opposite to the direction of change in the parameter. Increasing the parameter, making the weighting of one cost measure in the objective function of the efficiency problem more negative, decreases or leaves unchanged the optimal amount of that cost measure. This result is intuitively appealing. The weightings of the cost measures are negative because the decision-maker prefers less cost. If he has higher preference for less of a particular cost measure, if the weighting of the cost measure is more negative, then it is optimal to have less or the same amount of that cost measure. When two Ψ 's are changed no statement about the direction of change of the individual cost measures is possible. When one Ψ and one Φ are changed no statement about the direction of change of the individual benefit and cost measures is possible.

Study of the effect of changes in components of the technology matrices is important because these matrices are predictions about the future, made at the beginning of the planning period. It is of interest to know how the optimal solution to the efficiency problem is affected by an error in the predictions. However, the effect of a change in a component of one of the technology matrices is difficult to evaluate using the saddle point results. The components of the technology matrices are the parameters of the constraints. With two exceptions the components of the technology matrices are multiplied both by variables and by Lagrange multipliers. The effect of a change in a component of all but two types of technology matrices cannot be broken down into an effect on the variable or on the Lagrange multiplier. Only the effect on both together can be described.

Consider an increase in a component of the lower (input) part of a systems technology matrix, $\Delta s^{t}_{I+j,b}$. The inequality is:

$$\Delta s^{t}_{I+j,b}\ (\bar{\zeta}^{t}_{b}\hat{\lambda}^{(L)t}_{j} - \hat{\zeta}^{t}_{b}\bar{\lambda}^{(L)t}_{j}) \geqq 0$$

This is equivalent to:

$$\hat{\zeta}^{t}_{b}\hat{\lambda}^{(L)t}_{j}\ \Delta s^{t}_{I+j,b}\left(\frac{\bar{\zeta}^{t}_{b}}{\hat{\zeta}^{t}_{b}} - \frac{\bar{\lambda}^{(L)t}_{j}}{\hat{\lambda}^{(L)t}_{j}}\right) \geqq 0$$

If the j^{th} element contributes positive value to the objective function in time period t , $\hat{\lambda}^{(L)t}_{j} > 0$. If the

activity was operated at a positive level before the parameter change was made, $\hat{\zeta}_b^t > 0$. When these things are true:

$$\frac{\bar{\zeta}_b^t}{\hat{\zeta}_b^t} \geqq \frac{\bar{\lambda}_j^{(L)t}}{\hat{\lambda}_j^{(L)t}}$$

If the optimal $\lambda_j^{(L)t}$ increases with the parameter change $(\bar{\lambda}_j^{(L)t} > \hat{\lambda}_j^{(L)t})$, then the activity level must increase $(\bar{\zeta}_b^t > \hat{\zeta}_b^t)$ by a greater percentage. If the optimal $\lambda_j^{(L)t}$ decreases with the parameter change $(\bar{\lambda}_j^{(L)t} < \hat{\lambda}_j^{(L)t})$, then the activity level decreases by a smaller percentage or increases. Similar statements can be made about the effect of changing a component of the lower (input) part of a benefit technology matrix.

There is an added difficulty in analyzing the effect of a change in a component of a technology matrix which appears in the cost functions. The upper (output) parts of the production and systems technology matrices and the lower (input) parts of the production technology matrices appear in the domains of the cost functions. The change in the nonlinear cost functions caused by a change in a component of one of these technology matrices must be approximated by replacing the nonlinear C_m 's with linear approximations, $C_m{}^*$'s. The change in the C_m 's caused by a component change in one of the above-mentioned technology matrices is approximated by the change in the $C_m{}^*$'s caused by the component change.

Study of the effect of a change in a component of the lower (input) part of a production technology matrix illustrates the use of linear approximations to the cost functions. If $p^t_{J+k,g}$ is increased by $\Delta p^t_{J+k,g}$:

$$L^1(\bar{x}, \hat{\lambda}) - L(\bar{x}, \hat{\lambda}) \approx \sum_{m=1}^{M} \hat{\lambda}^{(C)}_m C_m{}^* (\Delta p^t_{J+k,g} \bar{\omega}^t_g)$$

$$L^1(\hat{x}, \bar{\lambda}) - L(\hat{x}, \bar{\lambda}) \approx \sum_{m=1}^{M} \bar{\lambda}^{(C)}_m C_m{}^* (\Delta p^t_{J+k,g} \hat{\omega}^t_g)$$

L^1 and L are identical for a change in $p^t_{J+k,g}$, except in the cost functions. Subtracting L from L^1 leaves only the change in the cost function, which is approximated by treating the cost functions as linear. Using the approximation the inequality is:

$$\sum_{m=1}^{M} \hat{\lambda}^{(C)}_m C_m{}^*(p^t_{J+k,g} \bar{\omega}^t_g) \geqq \sum_{m=1}^{M} \bar{\lambda}^{(C)}_m C_m{}^*(p^t_{J+k,g} \hat{\omega}^t_g)$$

The second Kuhn-Tucker condition says that the optimal $\lambda^{(C)}_m = -\Psi_m$ if the optimal $C_m > 0$. Therefore $\hat{\lambda}^{(C)}_m = \bar{\lambda}^{(C)}_m = -\Psi_m$ for all nonzero cost functions. Again using the approximation the inequality becomes:

$$-\sum_{m=1}^{M} \Psi_m C_m (\Delta p^t_{J+k,g} \bar{\omega}^t_g - \Delta p^t_{J+k,g} \hat{\omega}^t_g) \geqq 0$$

Since the $-\Psi$'s are negative, the optimal activity level must either decrease or remain unchanged when a component of the lower (input) part of the production technology matrix is increased.

Study of the effect of a change in a component of the upper (output) part of a production or systems technology matrix is very difficult. Not only must the cost functions

be approximated as linear functions, but Lagrange multipliers limit the statements about effects which can be made from the inequalities.

A change in a component in the upper (output) part of a benefit technology matrix, $\Delta b^t_{1,a}$, remains to be considered. The inequality is:

$$\hat{\lambda}_1^{(E)t} \Delta b^t_{1,a} \bar{\eta}^t_a \geqq \bar{\lambda}_1^{(E)t} \Delta b^t_{1,a} \hat{\eta}^t_a$$

The first Kuhn-Tucker condition says that optimal $\lambda_1^{(E)t} = \Phi_1^t$ if optimal $E_1^t > 0$. Therefore, if the benefit is produced:

$$\Phi_1^t \Delta b^t_{1.a} (\bar{\eta}^t_a - \hat{\eta}^t_a) \geqq 0$$

Since all Φ 's are positive the optimal activity level must increase or remain unchanged when a component in the upper (output) part of a benefit technology matrix is increased.

In this section the inequalities have been interpreted assuming increases in the changing parameters. The inequalities can also be interpreted assuming decreases in the changing parameters. The interpretations are the same, except that "decrease" replaces "increase" and vice versa.

IX. DECENTRALIZATION

The model represents the choice problem embedded in a PPB system. The efficiency problem includes all the information which is needed to calculate efficient alternatives. Nothing has been assumed, however, about where in the Department of Defense this information is known. It seems obvious that no centralized level of the Department of Defense has all this information. It has only been assumed that some real world procedure exists, such as the tatonnement process described in the Introduction, which allows decisions to be made as if all the information were known at a centralized level. The efficiency problem is therefore formulated as if all the information is known at a centralized level. It is of interest to see if the efficiency problem can be decomposed into a set of smaller problems. If a set of smaller problems which is equivalent to the efficiency problem can be found, then real world planning procedures need only solve the set of smaller problems. Intuitively this would be more efficient. Decomposing the efficiency problem into smaller subproblems is called decentralization.

One type of decentralization which bears close relation to the real world is decentralization by services. This would mean that the efficiency problem for the whole Department of Defense would be broken down into a set of

efficiency problems, one for each service. The monotonically increasing nature of the cost functions for the Department of Defense makes this type of decentralization difficult. Since the scale of operation of the whole Department of Defense determines costs, the cost functions of each service depend on the scale of operation of the other services. In order to decentralize by services planning procedures which handle this problem of externality of costs would have to be formulated.

Another type of decentralization is by submodel. The efficiency problem would be broken down into smaller, submodel problems. The submodel decision-maker's problem could be to select activity levels after he has received information from above and below. Suppose that the decision-maker receives imputed values or shadow prices for the inputs and outputs of his submodel. Suppose that the shadow prices are the optimal Lagrange multipliers from the original (now called consolidated) efficiency problem. Then any activity in the submodel which was operated at a positive level in the optimal solution to the consolidated problem will give zero profit when outputs and inputs are priced at their shadow prices. Any activity not used at a positive level in the consolidated problem will give zero or negative profit. This is due to the third, fourth, and fifth Kuhn-Tucker conditions. Assume that the submodel decision-maker's objective is to maximize profit when outputs and inputs of the submodel are priced at their shadow prices.

Then he will choose to operate at a positive level only activities which give zero profit because all other activities give negative profit. These activities include, but may not be restricted to, activities which were operated at a positive level in the consolidated problem. Since all activities give zero profit at any activity level he will set activity levels arbitrarily. There is no guarantee that he will choose the activity levels which were optimal in the consolidated problem. Thus the model cannot be decentralized by shadow prices alone.

Suppose that the decision-maker receives shadow prices for the submodel outputs from above, but from below he receives a vector of inputs which he must use to produce outputs. Suppose that his objective is to operate the activities at levels which use all the inputs and give maximum profits (no longer zero) when outputs are priced at their consolidated shadow prices. For a given time period let $\underline{\underline{T}}^{(O)}$ be the upper (output) part of the submodel technology matrix, $\underline{\underline{T}}^{(I)}$ the lower (input) part of the matrix, $\underline{\lambda}'$ the column vector of shadow prices of the outputs from the consolidated problem optimal solution, $\underline{\theta}$ the row vector of activity levels, and $\underline{c}'$ the column vector of inputs. Then the decision-maker's problem in the time period is:

$$\text{Max } \underline{\lambda}'\underline{\underline{T}}^{(O)}\underline{\theta} \qquad \text{subject to} \quad \underline{\underline{T}}^{(I)}\underline{\theta} = \underline{c}' \qquad \underline{\theta} \geqq \underline{0}$$

In the consolidated model the outputs of one submodel are the inputs to the next submodel. To specify the input vector of a submodel the outputs of the preceding submodel must be known. Therefore, this type of horizontal decentralization must be sequential. First profit in the production submodel must be maximized in each time period in the planning period using the element shadow prices and amounts of inputs which were optimal in the consolidated problem. Then profit in the systems submodels must be maximized in each time period using the consolidated characteristic shadow prices and the total amounts of elements produced in the production submodel in previous periods. Finally profit in the benefit submodel must be maximized using the consolidated program's shadow prices for measures of effectiveness and the amounts of characteristics produced in the systems submodel.

It has not been possible to prove that this sequential, horizontal decentralization gives the same solution to the efficiency problem as did the consolidated program. Investigation of special cases indicates that if shadow prices for outputs and the vector of inputs from the consolidated program are used, the optimal activity levels of the consolidated program will be optimal solutions to the submodel programs. However, the consolidated optimum is not a unique solution to the submodel program in special cases (e.g. when two or more activities operated at a positive level in the consolidated program use only a single input).

If activity levels other than the consolidated optimal levels are used in any submodel below the benefit submodel, then the input vectors to higher submodels will differ from the consolidated optimum and the decentralized solution will not match the consolidated. This method of decentralization requires that the resource input vector to the production submodel from the consolidated program be known.

Consider the validity of this sequential, horizontal decentralization as a conjecture. If the model can be decentralized in this way, then each submodel can be further decentralized. Assume that the technology matrix of a submodel in a time period describes the transformation of inputs into outputs in several different institutions. For example, assume that the Army and Navy can use inputs to produce outputs in the same ways: they have identical technology matrices. Let input vectors, $\underline{c}_A$ and $\underline{c}_N$, be given to the Army and Navy, respectively, where $\underline{c}_A + \underline{c}_N = \underline{c}_c$ and $\underline{c}_c$ is the vector of amounts of inputs optimally used in the consolidated program. Let the Army and Navy trade inputs at their optimal consolidated shadow prices, $\underline{\lambda}$. Let the input vectors after trading be $\underline{c}_A{}^*$ and $\underline{c}_N{}^*$. Assume that the Army and Navy maximize profits when outputs are priced at their shadow prices, using the after-trading vectors of inputs. The Army and Navy will choose activity levels $\underline{\theta}_A$ and $\underline{\theta}_N$, respectively, to solve the following linear programs:

$$\text{Max} \quad \underline{\lambda}'\underline{\underline{T}}^{(0)}\underline{\theta}_A \qquad \text{subject to} \qquad \underline{\underline{T}}^{(I)}\underline{\theta}_A = \underline{c}_A{*}' \qquad \underline{\theta}_A \geqq \underline{0}$$

$$\text{Max} \quad \underline{\lambda}'\underline{\underline{T}}^{(0)}\underline{\theta}_N \qquad \text{subject to} \qquad \underline{\underline{T}}^{(I)}\underline{\theta}_N = \underline{c}_N{*}' \qquad \underline{\theta}_N \geqq \underline{0}$$

Lancaster proves some statements which show that the sum of the activity level solutions to these linear programs is the same as the solution, $\underline{\theta}_c$, to the centralized submodel program.[8] The centralized submodel program is:

$$\text{Max} \quad \underline{\lambda}'\underline{\underline{T}}^{(0)}\underline{\theta}_c \qquad \text{subject to} \qquad \underline{\underline{T}}^{(I)}\underline{\theta}_c = \underline{c}_c' \qquad \underline{\theta}_c \geqq \underline{0}$$

This means that the decentralized submodel operates in the same way as the centralized submodel.

If the efficiency problem is decentralized sequentially and horizontally and then again by services, the resulting set of smaller problems appear to be easy to solve in the real world. It is only required that the services in the submodels know their own technology matrices.

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[8] Lancaster, Kelvin, Mathematical Economics, pp. 110-111. The Macmillan Company, New York, 1968.

X. SUMMARY

In this paper a model of the PPB problem has been formulated in which the variables are resources, elements, characteristics, benefits, and costs. An efficiency problem has been developed for the model. The efficiency problem has as its optimal solution an efficient PPB alternative. When all possible values of the parameters of the objective function of the efficiency problem are considered the optimal solutions constitute the PPB efficient frontier. "The plan" is one alternative on the efficient frontier, selected on the basis of the decision-maker's preferences. Decision rules which are necessary and sufficient for an efficient alternative have been derived. The discounting of benefits over time which is involved in choosing an alternative from the efficient frontier has been discussed. The effects of changes in the parameters of the model have been studied. Decentralization possibilities in the model have been considered.

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13. ABSTRACT

A model of the planning, programming, and budgeting problem is formulated. The variables of the model are resources, elements, characteristics, benefits (measures of effectiveness), and costs. The nature of the PPB problem requires that the model incorporate multiple measures of benefit and cost. To characterize efficient choices in the PPB context decision rules which are necessary and sufficient for efficiency are derived. Discounting of benefits over time is discussed. Sensitivity analysis of the model is performed. Decentralization possibilities in the model are explored.

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